Useful discussions with L. De Maeyer and the fine cooperation of J. Hendrix are deeply appreciated.

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Divergence of high order Gaussian modes

William B. Bridges

California Institute of Technology, Pasadena, California 91125.

Received 28 June 1975.

A simple expression for the beam divergence from a laser oscillating in a high order Gaussian mode can be derived from the properties of the mathematical expressions for these modes.[1] For simplicity, we first consider a symmetric confocal resonator of spacing b, which contains a laser medium of diameter $2a$, as pictured in Fig. 1. The intensity distribution at a distance z from the center of the resonator will have the form

$$I(r, z, \phi) \sim \text{polynomial}\left(\frac{r}{w}\right) \cdot f(z) \exp\left(-2\frac{r^2}{w^2}\right)\left(\begin{matrix}\sin l\phi \\ \cos l\phi\end{matrix}\right) \quad (1)$$

or

$$I(x, y, z) \sim \text{polynomial}\left(\frac{x}{w}\right) \times \text{polynomial}\left(\frac{y}{w}\right) f(z) \exp\left(-2\frac{x^2}{w^2} - 2\frac{y^2}{w^2}\right), \quad (2)$$

where

$$w^2(z) = w_o^2\left[1 + \left(\frac{\lambda z}{\pi w_o^2}\right)^2\right] \quad (3)$$

and

$$w_o^2 = \frac{b\lambda}{2\pi}. \quad (4)$$

The essential property of Eqs. (1) and (2) of interest here is that the transverse coordinate always occurs in ratio with the auxiliary function $w(z)$. For the lowest order TEM_{00} mode, the polynomials in Eqs. (1) or (2) become unity; and $w(z)$ traces the locus of the exp(−2) intensity points, usually taken as the diameter of that mode.

For higher order modes, a different definition for diameter is required. We could choose, for example, the largest radius r at which the intensity is exp(−2); or, alternatively, the radius of the largest zero of the polynomial; or some other such description. However the diameter is defined, it will be defined as the radius to some *feature* on the distribution; call this radius $r_f(z)$. It follows from the form of Eqs. (1)–(3) that the ratio r_f/w for a particular feature is invariant with z. Thus

$$\frac{r_f(0)}{w_o} = \frac{r_f(b/2)}{w(b/2)} = \frac{r_f(z)}{w(z)}. \quad (5)$$

If we have made a reasonable choice of the feature to which we have defined the diameter, then we can also assume that the laser medium diameter $2a$ will select a high order mode of diameter $2r_f(b/2)$; that is, a mode that just fills the medium. If we further note that for a confocal resonator $w(b/2) = w_o\sqrt{2}$, then from Eq. (5) we may write

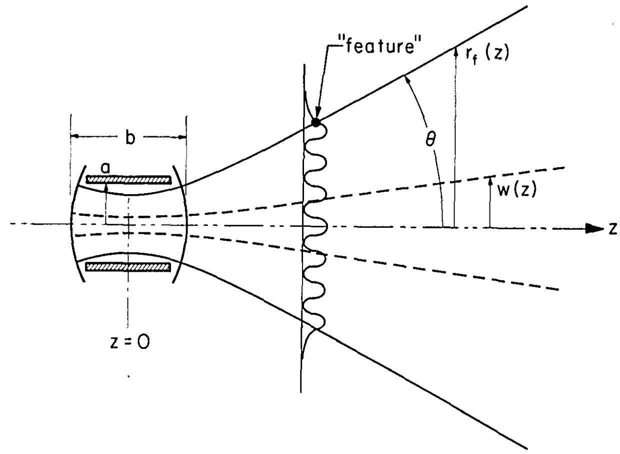

Fig. 1. Coordinate system for a high order mode laser with a confocal cavity of spacing b, mirror radius b, and medium diameter $2a$. The high order mode diverges with a half-angle θ, measured to some defined feature on the mode intensity distribution.

$$\frac{a}{\sqrt{2}w_o} = \frac{r_f(z)}{w(z)}. \quad (6)$$

Defining the divergence half-angle θ as

$$\theta \equiv \lim_{z\to\infty} \frac{r_f(z)}{z} \quad (7)$$

and noting that for large z

$$w(z) \simeq \frac{\lambda z}{\pi w_o}, \quad (8)$$

we have

$$\theta = \frac{\lambda a}{\sqrt{2}\pi w_o^2} \quad (9)$$

But $w_o^2 = b\lambda/2\pi$, so

$$\theta = \sqrt{2}(a/b). \quad (10)$$

Equation (10) is the simple expression sought for the beam divergence of a high order mode from a confocal cavity of diameter $2a$ and spacing b. Equation (10) states that the divergence is essentially equal to the geometric aspect ratio of the laser medium, independent of wavelength and mode number. A longer wavelength laser in the same cavity would, of course, oscillate in a lower order mode; but the divergence half-angle would be the same.

If the mirror radii R are not equal to the spacing b, then Eq. (10) becomes

$$\theta = \sqrt{2}(a/b)\cdot(b/R)^{1/2}, \quad (11)$$

still independent of λ and the indices of the particular mode oscillating.

As R is increased from the confocal value b, both the mode index and the divergence decrease. The assumptions made in the analysis break down when R is finally made so large that the radius of the feature chosen as the diameter becomes larger than the diameter $2a$ for even the TEM_{00} mode. Radii $R \leq b/2$, corresponding to unstable resonators, are also not allowed since a division by $(2R - b)$ was made in obtaining Eq. (11). And, of course, all bets are off if some other phenomenon such as nonuniform radial gain distribution or saturation acts to select the highest order mode rather than the geometric medium diameter $2a$.

Figure 2 compares the theoretical expression, Eq. (11), with experimental data[2] taken with a relatively large diameter hollow-cathode Hg^+ laser,[3] $\lambda = 0.6150\ \mu m$. This particular laser employed a flat output mirror and a 100-cm radius high reflectance mirror, spaced at 66 cm. A mode-selecting iris was located at the flat mirror so a = aperture diameter/$\sqrt{2}$, $R = 100$ cm, and $b = 132$ cm for the equivalent symmetrical resonator. The agreement is excellent for the higher order modes (mode index > 10).

It is also interesting to consider another simple expression that follows from Eq. (10). Suppose we treat a pth order radial mode as dividing the radius at the beam waist $a/\sqrt{2}$ into p phase-reversed regions of extent $a/p\sqrt{2}$. (In actuality, the zeros of the appropriate polynomials are *not* exactly equally spaced, but the assumption that they are is a reasonable first approximation.) Suppose further that the beam divergence θ is determined by the size of this small region

$$\theta \simeq \frac{\lambda}{a/p\sqrt{2}} . \tag{12}$$

Equating Eqs. (10) and (12) yields a simple expression for the mode index p:

$$p \simeq \frac{a^2}{b\lambda} \equiv \text{Fresnel number}. \tag{13}$$

Equation (13) may be compared with the actual behavior of the polynomial solution. If the zeros of the first fifteen Laguerre polynomials[4] are plotted and an asymptotic expression fitted by eye, the relation

$$\frac{r_f(0)}{w_o} \cong 1.4\ p^{1/2} \tag{14}$$

results, provided we take the feature to be the radius to the largest zero. Assuming the aperture radius will limit the mode at this feature, $a = (2)^{1/2} r_f(0)$ gives

$$p \cong 1.6 \frac{a^2}{b\lambda}, \tag{15}$$

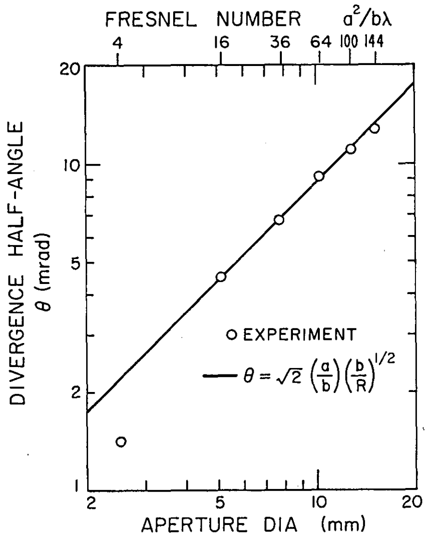

Fig. 2. Comparison of experimentally measured divergence of high order modes and the simple expression given in Eq. (11). The cavity Fresnel number is also given along the top of the graph.

a reasonable agreement with Eq. (13). If we were to take a more reasonable feature, such as some point on the outer tail as indicated schematically in Fig. 1, then Eq. (15) would be even closer to Eq. (13).

It seems that the crude picture of the highest order mode as two plane waves intersecting at an angle $[(2)^{1/2}a/b]$ predicts both θ and p reasonably well, in contradiction to the statement by Abramskii that it does not.[5]

If useful for nothing else Eq. (10) will allow you to mystify your friends when you visit their laboratory by telling them the divergence of their new laser before they even tell you what the active medium is.

The author would like to thank the Sherman Fairchild Foundation for support while this work was being done. His permanent address is Hughes Research Laboratories, Malibu, California 90265.

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Graphical aid to the transformation of a laser beam by a thin lens

J. Svetlik

Physics Department, Faculty of Engineering, Kravi hora XI, Brno, Czechoslovakia.
Received 2 July 1974.

The diagram proposed here is especially convenient for quick preliminary design of lens systems that use laser beams. It enables one to examine a number of variants in a short time before adopting and calculating the optimum variant by either the use of well-known formulas[1,2] or a graphical method.[3]

An ideal thin lens of focal length f transforms an incoming fundamental Gaussian beam with the confocal parameter $b_1 = 2\pi w_1^2/\lambda$ into a Gaussian beam with the confocal parameter $b_2 = 2\pi w_2^2/\lambda$. The beam waists, W_1 and W_2, of these two beams are located at distances d_1 and d_2 from the lens (Fig. 1). The parameters b_2 and d_2 are related to the parameters b_1 and d_1 by

$$(d_1 - f)/(d_2 - f) = b_1/b_2, \tag{1}$$

and

$$(d_1 - f)(d_2 - f) = f^2 - \frac{1}{4} b_1 b_2, \tag{2}$$

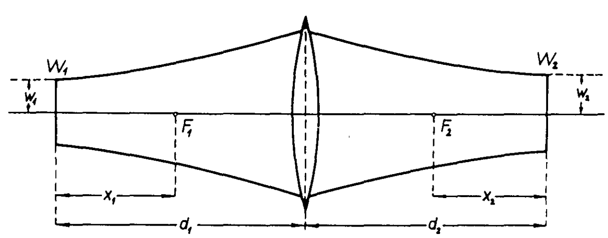

Fig. 1. Transformation of a laser beam by a thin lens.